Journal of Global Optimization **18:** 195–210, 2000.

Solving Nonconvex Planar Location Problems by Finite Dominating Sets

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(Received 23 March 2000; accepted in revised form 19 April 2000)

Abstract. It is well-known that some of the classical location problems with polyhedral gauges can be solved in polynomial time by finding a finite dominating set, i.e. a finite set of candidates guaranteed to contain at least one optimal location.

In this paper it is first established that this result holds for a much larger class of problems than currently considered in the literature. The model for which this result can be proven includes, for instance, location problems with attraction and repulsion, and location-allocation problems.

Next, it is shown that the approximation of general gauges by polyhedral ones in the objective function of our general model can be analyzed with regard to the subsequent error in the optimal objective value. For the approximation problem two different approaches are described, the sandwich procedure and the greedy algorithm. Both of these approaches lead - for fixed ϵ - to polynomial approximation algorithms with accuracy ϵ for solving the general model considered in this paper.

Key words: Continuous Location, Polyhedral Gauges, Finite Dominating Sets, Approximation, Sandwich Algorithm, Greedy Algorithm

1. Introduction

In recent years, research in location theory has been very active in models, which can be solved using finite dominating sets (FDS), i.e. a set of finite cardinality which contains an optimal location for the respective problem. If, in addition, the cardinality of the FDS is polynomial in the input of the location problem the FDS approach yields a polynomial solution algorithm, even in the worst case where an extensive search for all candidates in the FDS is performed.

A predecessor of this idea is the median algorithm (see, for instance, [9, 24, 10]) for solving $1/P/\bullet/l_1/\sum^\star$, i.e. the problem of finding a best location x in the

* Partially supported by grant PB96-1416-C02-02 of Dirección General de Enseñanza Superior, Spain.

⋆ In this paper we use the 5-position classification scheme for location problems of [13] where Positions 1 through 5 characterize number and type of the new facility(ies), the environment (e.g., plane such that the sum of the weighted rectilinear distances

$$F(x) := \sum_{k=1}^{N} \omega_k \|x - a_k\|_1 \tag{1.1}$$

of x to the existing facilities $a_1, \dots, a_N \in \mathbb{R}^2$ is minimized. The FDS consists of the grid points given by the intersection of the rectilinear grid lines passing through $a_1, \dots, a_N$.

A generalization of the previous problem is obtained if the rectilinear distance $\|x - a_k\|_1$ is replaced in (1) by a polyhedral gauge (see, e.g. [25, 29, 26])

$$\gamma_{B_k}(x) := \inf\{\lambda > 0 : \frac{1}{\lambda} x \in B_k\} \tag{1.2}$$

where B_k is for each $k = 1, \dots, N$ a convex polytope in $\mathbb{R}^2$ containing the origin in its interior. An FDS for the resulting location problem $1/P/\bullet/\gamma_{pol}/\sum$ is given by the grid points of the grid defined by the rays starting in a_k and passing through each of the extreme points of $B_k, k = 1, \dots, N$ (see [8] and – in more general form – Theorem 1). Obviously, this FDS is polynomial in the size of the problem, if the input is N, the number of existing facilities, and V, the maximal number of extreme points in any of the polytopes B_k.

Additional polynomial FDSs have been found for restricted problems $1/P/\mathcal{R}/\gamma_{pol}/\sum$ where a region $\mathcal{R}$ is excluded from siting new locations ([12, 26, 20, 14], for barrier problems $1/P/\mathcal{B}/\gamma_{pol}/\sum$, where additionally trespassing is forbidden [11, 6, 22], and for ordered Weber problems $1/P/\bullet/\gamma_{pol}/\sum_{ord}$, a class of problems including – among others - sum and maximizing objectives [30, 31].

The common feature of the problems in which the FDS approach has been applied successfully is the fact that distances are of the type (1.2), where B_k is a polytope.

The goal of this paper is to show, that the FDS approach carries much further than that. In Section 2, we introduce a very general location model which includes problems with attraction and repulsion, location-allocation and gauges defined by arbitrary compact, convex sets (i.e. non necessarily polytopes). Problems of this type can so far only be tackled by standard methods of global optimization, which do not use the specifics of the location background. In Section 3 we show, that this problem class can be solved with any required accuracy by reducing it to a problem solvable with an FDS approach. Sections 4 and 5 contain two proposals how the general reduction idea can be specified using the sandwich approximation technique of [3] and a greedy approach, respectively. The paper is concluded by a summary of the results and a list of further research projects which are stimulated by the ideas of this paper.

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planar, network, discrete), specialties (restrictions, barriers, constant weights, etc.), distance functions, and type of objective function (sum, max, multi-objective, etc.), respectively. Here, bullets indicate unspecified items.

2. The location model – formulation and examples

In this paper, we consider a general planar location problem in which N points $a_1, \dots, a_N$ are given representing the geographic coordinates of demand points or existing facilities. The distance between facilities is measured using N gauges $\gamma_{B_1}, \dots, \gamma_{B_N}$ defined by their respective unit balls $B_1, \dots, B_N$ which are compact, convex sets with the origin in their interior. Hence, the gauges are defined by

$$\gamma_{B_k}(x) := \inf\{\lambda > 0 : \frac{1}{\lambda}x \in B_k\} \tag{2.3}$$

Note that 1.2 is a special case of this definition in which the unit balls B_k are convex polytopes for all $k = 1, \dots, N$. (See e.g. [25, 29] for further properties on gauges.)

The problem addressed in this paper can be written as

$$\min_{x\in S} F(x) := \Phi(x, \gamma_{B_1}(x - a_1), \gamma_{B_2}(x - a_2), \dots, \gamma_{B_N}(x - a_N)), \tag{2.4}$$

where $\Phi : \mathbb{R}^2 \times \mathbb{R}^N_+ \longrightarrow \mathbb{R}$ satisfies

1. S is a bounded polygonal region in $\mathbb{R}^2$.
2. For each $x \in S$, the function $\Phi(x, \cdot) : \mathbb{R}^N_+ \longrightarrow \mathbb{R}$ is componentwise nondecreasing, i.e., if any $u := (u_1, \dots, u_N)$, $v := (v_1, \dots, v_N)$ are such that $u_i \leqslant v_i$ for all $i = 1, \dots, N$, then
$$\Phi(x, u) \leqslant \Phi(x, v)$$
3. For any $x \in S$, the function $\Phi(x, \cdot)$ is Lipschitz-continuous with Lipschitz constant $L > 0$, i.e., for all $x \in S$ and all $u, v \in \mathbb{R}^N_+$,
$$|\Phi(x, u) - \Phi(x, v)| \leqslant L\|u - v\|,$$
where $\|\cdot\|$ denotes the Euclidean norm.
4. Φ is quasiconcave, i.e., the sets $\{(x, u) : \Phi(x, u) \geqslant \alpha\}$ are convex for all α, [1].

If we want to emphasize the dependence of the objective function on the chosen unit balls we sometimes write the objective function as $F_{B_1,\dots,B_N}$ or as F_B.

The model under consideration is general enough to include as particular cases many notoriously difficult planar single-facility location problems encountered in the literature. Some examples are listed below.

The *Weber problem with attraction and repulsion* can, for instance, be written as

$$\Phi(x, u) = -\int \gamma_B(x - c)\, \mathrm{d}\mu(c) + \sum_{k=1}^{N} \omega_k u_k,$$

where ω_k, $1 \leqslant k \leqslant N$ are nonnegative weights, μ is a measure in the plane and γ_B is a gauge such that, for each x, $\gamma_B(x - \cdot)$ is integrable with respect to μ.

Then, (2.4) becomes

$$\min_{x\in S}\sum_{k=1}^{N}\omega_k\gamma_{B_k}(x-a_k)-\int\gamma_B(x-c)\,\mathrm{d}\mu(c). \tag{2.5}$$

In case μ has only mass on a finite set of points $c_1,\ldots,c_T$, (2.5) has the more familiar form

$$\min_{x\in S}\sum_{k=1}^{N}\omega_k\gamma_{B_k}(x-a_k)-\sum_{t=1}^{T}\mu(\{c_t\})\gamma_B(x-c), \tag{2.6}$$

addressed e.g. in [5, 7, 32, 35].

Another class of problems which is covered by our general approach are *location-allocation Weber problems*. Within this category, we may consider single-facility location problems in which the locational decision yields also allocation of demand, see [29]. A relevant instance is the so-called *Profit-maximizing Weber problem*, [17, 27]: Set

$$\Phi(x,u)=\min_{(\pi_1,\ldots,\pi_N)\in\Pi}\left(\sum_{k=1}^{N}\pi_k\left(g_k(u_k)-D_k(\pi_k)\right)\right),$$

where Π is a compact subset of $\mathbb{R}_+^N$, $g_1,\ldots,g_N:\mathbb{R}_+\longrightarrow\mathbb{R}$ are concave non-decreasing functions with directional derivative bounded, and $D_1,\ldots,D_N$ are arbitrary functions. Then, Problem (2.4) can be written as

$$\min_{x\in S,\pi\in\Pi}\sum_{k=1}^{N}\pi_k\left(g_k(\gamma_{B_k}(x-a_k))-D_k(\pi_k)\right),$$

or equivalently as

$$\max_{x\in S,\pi\in\Pi}\sum_{k=1}^{N}\pi_k\left(D_k(\pi_k)-g_k(\gamma_{B_k}(x-a_k))\right), \tag{2.7}$$

which has the following interpretation: together with the locational decision, we can chose the prices $(\pi_1,\ldots,\pi_N)\in\Pi$ charged per unit of product delivered to the demand points (markets) $a_1,\ldots,a_N$. The demand of market a_k if price π_k is charged is $D_k(\pi_k)$, and the transportation cost per unit of product are given by a function g_k of the distance $\gamma_{B_k}(x-a_k)$ separating the facility and a_k. Hence, the total profit, to be maximized, is given by the objective function of (2.7).

An important particular case (although the allocation part disappears), is given by the choice $\Pi=\{\pi^0\}$, i.e., the price is not any more a decision variable. Then, up to irrelevant additive constants, Φ has the form $\sum_{k=1}^{N}\pi_k^0 g_k(u_k)$, thus Problem (2.4) becomes the Weber problem in which costs are assumed to be non-decreasing concave (not necessarily affine) functions of distances, [19, 34].

Other single-facility location-allocation easily accommodated within our framework are *Weber problem with supply surplus* introduced in [21] or the Weber problem with alternative transportation systems, addressed in [4].

Obviously, the objective function F of Problem (2.4) is not differentiable. Moreover, F is a composition of a nondecreasing quasiconcave and convex functions $(\gamma_{B_k}(\cdot - a_k))$, in general neither (piecewise) convex nor concave, such that several local optimal solutions may exist which are not globally optimal. Hence, if global optimal solutions are sought, one is obliged to use Global-Optimization procedures, from the simplest grid-search to more sophisticated techniques, such as polyhedral annexation, outer approximation or branch and bound schemes ([15, 18]).

These methods may be successfully applied if the feasible region S is a rectangle or a convex polygon, but if S does not have a nice shape, these general-purpose techniques may be hard to implement (think, for instance, of the construction of a regular covering grid over a nonconvex polygon) or may not work at all (e.g. the polyhedral-annexation procedure described in [34]). Only (variants of) the branch-and-bound method introduced in [16] seem to be of use for these cases, see [15, 28]. Even if in some cases the latter methods may work well in practice, it is only known that, for any accuracy $\varepsilon > 0$, they stop after a finite number of iterations yielding an ε-optimal solution, [28]. However, no worst-case analysis, providing the order of magnitude of such finite number of iterations to obtain an ε-optimal solution seems to have been described so far.

In the following section we will therefore introduce an approach which – under the rather weak assumptions of our model – results in algorithms with a priori known complexity bounds.

3. Approximation results

We first analyze Problem 2.4 in the case of polyhedral gauges.

If the unit balls B_k are polytopes, we consider for each $k = 1, \dots, N$ the cones defined by the rays starting from a_k in the direction of the corner points of B_k. It is well-known (see [8, 25, 29]) that γ_{B_k} defined by (2.3) is a linear function in each of these cones. Hence, their intersection over all $k = 1, \dots, N$ defines a tessellation of the plane into polyhedra such that within these polyhedra each of the functions $\gamma_{B_k}(x - a_k)$ is affinely linear in x.

For an arbitrary set X we denote by $\text{ext}(X)$ the set of all extreme points of the convex hull $\text{conv}(X)$ of X. Since a concave function attains its minimum at extreme points of the feasible region, the discussion above implies the following result.

THEOREM 1. *Suppose $B_1, \dots, B_N$ are polytopes. Let $\{P_i,\ i \in I\}$ be a finite set of polyhedra covering $\mathbb{R}^2$ such that each $\gamma_{B_k}(x - a_k)$ is affinely linear in x within each P_i. Then, the set $\{ext(S \cap P_i),\ i \in I\}$ is an FDS for problem (2.4) (see Figure 1).*

Proof. Since $\{P_i,\ i \in I\}$ covers the plane, for any $x \in S$ there exists $i^* \in I$ such that x belongs to the polygon $P_{i^*} \cap S$.

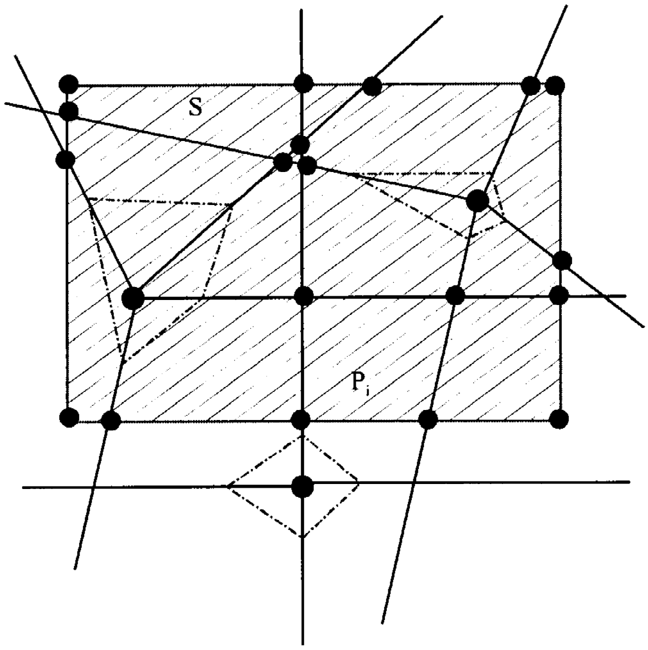

Figure 1. Illustration for Theorem 1.

Since each $\gamma_{B_k}(x-a_k)$ is affinely linear within P_{i^*}, it follows that, within P_{i^*}, F is the composition of the quasiconcave function Φ with the affinely linear function mapping $y \in P_{i^*}$ to $(y, \gamma_{B_1}(y-a_1), \ldots, \gamma_{B_N}(y-a_N))$. Thus F is quasiconcave on $\mathrm{conv}(P_{i^*} \cap S)$, and, therefore, attains its minimum on $\mathrm{conv}(P_{i^*} \cap S)$ in some element of $\mathrm{ext}(P_{i^*} \cap S)$.

Hence, any feasible x is dominated by some $x^* \in \bigcup_{i\in I} \mathrm{ext}(P_{i^*} \cap S)$, as asserted. □

As a consequence of Theorem 1, Problem (2.4) can – for polyhedral gauges – be reduced to inspecting the finite list of points in the set $\bigcup_{i\in I} \mathrm{ext}(P_i \cap S)$. If each B_k has $v(B_k)$ extreme points, $V := \max_{k=1}^{N} v(B_k)$ and S has $v(S)$ extreme points, then $\bigcup_{i\in I} \mathrm{ext}(P_i \cap S)$ will have cardinality

$$O(N^2V^2 + NVv(S)), \tag{3.8}$$

and can be constructed by well-known computational geometry techniques [2, 23].

In the case of gauges defined by arbitrary compact, convex unit balls $B_1, \ldots, B_N$, Theorem 1 obviously does not hold. The best we can hope for in this general situation is an FDS result for an approximate solution of Problem (2.4).

The following, straightforward property will be of use to achieve this goal.

THEOREM 2. *Given two compact convex sets B_1, B_2 with the origin in the respective interior, the following property holds.*

1. *If* $B_1 \subseteq B_2$ *then* $\gamma_{B_1} \geqslant \gamma_{B_2}$
2. *For any* $\Delta > 0$, $\gamma_{(\Delta B)} = \frac{1}{\Delta}\gamma_B$

Next, we will use the Lipschitz property of the function Φ to see the impact for Problem (2.4) of replacing unit balls in the definition of gauges by other unit balls and prove the following approximation result. (Recall, that $F_Q(x)$ stands for the objective function $F(x)$ in Problem (2.4) in which the gauges are defined with respect to the set $Q = \{Q_1, \ldots, Q_N\}$ of unit balls.)

THEOREM 3. *Let* C_k, E_k, D_k *be compact, convex sets with* $0 \in int(C_k)$ *and*

$$C_k \subseteq E_k \subseteq D_k \subseteq \Delta C_k, \qquad k = 1, \ldots, N, \tag{3.9}$$

for some $\Delta \geqslant 1$. *Moreover, let* $L > 0$ *be the given Lipschitz constant for* $\Phi(x, \cdot)$, *and let* M *satisfy*

$$M \geqslant \max_{y \in ext(S)} \|\gamma_{C_1}(y - a_1), \ldots, \gamma_{C_N}(y - a_N)\| \tag{3.10}$$

Then we get for any $P, Q \in \{C, D, E\}$ *and* $\varepsilon = L(1 - \frac{1}{\Delta})M$

1. $$0 \leqslant |F_P(x) - F_Q(x)| \leqslant \varepsilon \tag{3.11}$$
2. *Any optimal solution* x_P *for* $\min_{x \in S} F_P(x)$ *is an* ε*-optimal solution for* $\min_{x \in S} F_Q(x)$

Proof. By Theorem 2,

$$\gamma_{C_k}(x) \geqslant \gamma_{E_k}(x) \geqslant \gamma_{D_k}(x) \geqslant \frac{1}{\Delta}\gamma_{C_k}(x) \qquad \forall x \in S,\ k = 1, 2, \ldots, N$$

Since, by assumption, $\Phi(x, \cdot)$ is componentwise nondecreasing,

$$\begin{aligned}
F_C(x) &= \Phi(x, \gamma_{C_1}(x - a_1), \ldots, \gamma_{C_N}(x - a_N)) \\
&\geqslant \Phi(x, \gamma_{E_1}(x - a_1), \ldots, \gamma_{E_N}(x - a_N)) \\
&\geqslant \Phi(x, \gamma_{D_1}(x - a_1), \ldots, \gamma_{D_N}(x - a_N)) \\
&\geqslant \Phi(x, \frac{1}{\Delta}\gamma_{C_1}(x - a_1), \ldots, \frac{1}{\Delta}\gamma_{C_N}(x - a_N))
\end{aligned}$$

Hence for all $P, Q \in \{C, D, E\}$

$$\begin{aligned}
0 &\leqslant |F_P(x) - F_Q(x)| \\
&\leqslant F_C(x) - \Phi(x, \frac{1}{\Delta}\gamma_{C_1}(x - a_1) \ldots, \frac{1}{\Delta}\gamma_{C_N}(x - a_N)), \\
&\leqslant L\|(\gamma_{C_1}(x - a_1), \ldots, \gamma_{C_N}(x - a_N)) - \frac{1}{\Delta}(\gamma_{C_1}(x - a_1), \ldots, \\
&\quad \gamma_{C_N}(x - a_N))\| \\
&= L(1 - \frac{1}{\Delta})\|(\gamma_{C_1}(x - a_1), \ldots, \gamma_{C_N}(x - a_N))\| \\
&\leqslant L(1 - \frac{1}{\Delta}) \max_{y \in S} \|(\gamma_{C_1}(y - a_1), \ldots, \gamma_{C_N}(y - a_N))\|
\end{aligned}$$

Since the function mapping $u \in \mathbb{R}^N_+$ to $\|u\|$ is convex and componentwise increasing, and the function assigning to each $x \in \mathbb{R}^2$ the value $(\gamma_{C_1}(x - a_1), \dots, \gamma_{C_N}(x - a_N))$ is convex, it follows that the function $x \in \mathbb{R}^2 \longmapsto \|(\gamma_{C_1}(x - a_1), \dots, \gamma_{C_N}(x - a_N))\|$ is also convex, thus attaining its maximum on S at some point in ext(S). In other words,

$$\begin{aligned}\max_{y\in S} \|(\gamma_{C_1}(y - a_1), \dots, \gamma_{C_N}(y - a_N))\| &= \max_{y\in \mathrm{ext}(S)} \|(\gamma_{C_1}(y - a_1), \dots, \\ &\qquad \gamma_{C_N}(y - a_N))\| \\ &\leqslant M,\end{aligned}$$

such that

$$|F_P(x) - F_Q(x)| \leqslant L(1 - \frac{1}{\Delta})M,$$

as claimed in (3.11).

If x_P and x_Q denote optimal solutions for $\min_{x\in S} F_P(x)$ and $\min_{x\in S} F_Q(x)$, respectively, then we get

$$\begin{aligned}|F_P(x_P) - F_Q(x_Q)| &= F_P(x_P) - F_Q(x_Q)(\text{ if not, interchange P and Q}) \\ &\leq F_P(x_Q) - F_Q(x_Q) \\ &\leq L(1 - \frac{1}{\Delta})M,\end{aligned}$$

showing the $\varepsilon = L(1 - \frac{1}{\Delta})M$- optimality. □

Theorem 3 enables us to solve Problem 2.4 with any required accuracy ε by choosing $P = B = \{B_1, \dots, B_N\}$ as the originally given unit balls B of Problem 2.4 and $Q = \tilde{B} = \{\tilde{B}_1, \dots, \tilde{B}_N\}$ as a set of polyhedral balls satisfying (3.9) according to the following algorithm.

ALGORITHM 1. *(Input: $\varepsilon > 0$; Output: $\tilde{x}$, ε-optimal for Problem (2.4)).*

Step 0: Set

$$\begin{aligned}M &= \max\{\frac{\varepsilon}{L}, \max_{y\in \mathrm{ext}(S)} \|\gamma_{B_1}(y - a_1), \dots, \gamma_{B_N}(y - a_N)\|\} \\ \Delta &= 1 + \frac{\varepsilon}{LM - \varepsilon}\end{aligned}$$

Step 1: Find a set $\tilde{B}$ of polytopes $\tilde{B}_1, \dots, \tilde{B}_N$ such that $P = \tilde{B}$ and $Q = B$ satisfy the conditions of Theorem 3.

Step 2: Use Theorem (1) to find an optimal solution $\tilde{x}$ of

$$\min_{x\in S} \tilde{F}(x) := \Phi(x, \gamma_{\tilde{B}_1}(x - a_1), \gamma_{\tilde{B}_2}(x - a_2), \dots, \gamma_{\tilde{B}_N}(x - a_N)). \tag{3.12}$$

STOP $\tilde{x}$ satisfies

$$|F_{\tilde{B}}(\tilde{x}) - \min_{x \in S} F(x)| \leq \varepsilon$$

As we have seen in the beginning of this section, Step 2 can be done for fixed ε in polynomial time, where the complexity of this step is depending on the maximal number V of extreme points in any of the polytopes $\tilde{B}_k$. It is therefore crucial to choose the polytopes in such a way that V is as small as possible. In the subsequent section, we will present two approaches dealing with this problem. In the first approach based on the sandwich algorithm of [3, 33] we choose in Theorem 3 $E_k = B_k$ and C_k and D_k as inner and outer approximation of B_k, respectively. The resulting algorithm will produce an a priori bound on the cardinality of a FDS to solve Problem 2.4 with required accuracy ε. In the second approach, a Greedy procedure is applied to find a so-called polyhedral, convex separator E_k separating $C_k = B_k$ and $D_k = \Delta B_k = (1 + \frac{\varepsilon}{LM-\varepsilon})B_k$. It will be shown that the number of extreme points produced with this procedure is at most by 1 larger than the smallest possible one.

4. Finding approximating polytopes by the sandwich procedure

We use the sandwich algorithm proposed by [3] for univariate convex functions and applied by [33] for approximation of convex bodies. The idea of the sandwich approach is to iteratively approximate a given convex body B with the goal of getting at the end of the iterations a required accuracy $\delta \geqslant 0$ by an interior polyhedron B^i and an outer polyhedron B^o, respectively, i.e. $B^i \subseteq B \subseteq B^o \subseteq (1+\delta)B$. In each iteration of the algorithm we check whether the Hausdorff distance (with respect to Euclidean distance l_2)

$$H_{l_2}(B^i, B^o) = \max_{x \in B^o} \min_{y \in B^i} l_2(x, y) > \delta.$$

If this is not the case, $H_{l_2}(B^i, B^o) \leqslant \delta$ as required. Otherwise, a pair $x \in B^o$, $y \in B^i$ with $l_2(x, y) > \delta$ is chosen and the point $z \in [x, y] \cap bd(B)$ is identified. B^i and B^o are updated by choosing z in the inner polyhedron B^i as additional extreme point and in the outer polyhedron B^o as supporting point (see Figure 2).

If R is the circumference of B, it can be shown (see [3, 33]) that no more than

$$\max \left\{ 4, \sqrt{\frac{8R}{\delta}} + 2 \right\}$$

many iterations are needed before the procedure stops with $B^i \subseteq B \subseteq B^o$ such that $H_{l_2}(B^i, B^o) \leqslant \delta$.

Since B is sandwiched by B^i and B^o we obtain the same bound for the Hausdorff distances between B and B^i, and B and B^o, i.e.

$$H_{l_2}(B, B^o) \leqslant \delta \qquad \text{and} \qquad H_{l_2}(B, B^i) \leqslant \delta.$$

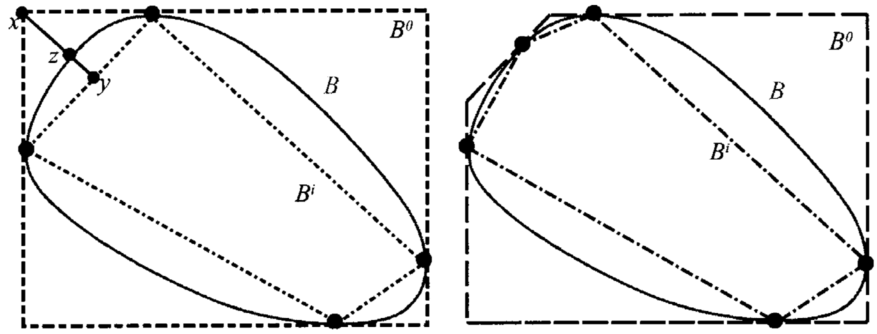

Figure 2. One iteration of the sandwich algorithm. The distance $l_2(x, y) > \delta$ such that z is included as additional point in both approximating polyhedra.

Consequently, both, B^i and B^o, can be used as δ-approximation of B. We can therefore define in Theorem 3 $C_k = B_k^i, E_k = B_k$ and $D_k = B_k^o$ for all $k = 1, \ldots, n$, and $\Delta = \delta + 1$. If R_k is the circumference of B_k, $k = 1, \ldots, N$ and V_k an integer such that $V_k \geqslant \max\left\{4, \sqrt{\frac{8 \cdot R_k}{\delta+1}} + 2\right\}$, then B_k^i and B_k^o are polyhedra with $O(V_k)$ many extreme points. Using $V := \max_k V_k$, the complexity result for solving location Problem (2.4) with respect to polynomial gauges, (see Theorem 1 and the subsequent analysis) and the interrelation $\delta = \Delta - 1 = \frac{\epsilon}{L\mu - \epsilon}$, Theorem 3 yields the following result.

THEOREM 4. *An ε-optimal solution of Problem 2.4 can be obtained by considering an FDS of cardinality $O(N^2V^2 + Nv(S)V)$ where $v(S)$ is the number of extreme points of the feasible region S.*

5. Finding approximating polytopes by the Greedy algorithm

In this section we will separate the closed convex sets $C_k = B_k$ and $D_k = \Delta B_k = (1 + \frac{\varepsilon}{LM - \varepsilon})B_k$ by a polytope E_k in the sense of Theorem 3. A polytope E_k with the required property $C_k \subseteq E_k \subseteq D_k$ is called a convex separator with respect to C_k and D_k, denoted $cs(D_k \setminus C_k)$. In order to simplify the denotation we will in the following delete the index k and investigate the problem of finding for given closed, convex sets C and D with $C \subseteq D$ a convex separator $cs(D \setminus C)$, i.e. $C \subseteq cs(D \setminus C) \subseteq D$

The boundary $bd(cs(D \setminus C))$ of $cs(D \setminus C)$ is a closed polygonal curve. If the context is clear we often call the boundary itself a convex separator. Our goal is to find convex separators with the smallest possible number of extreme points, a *minimum convex separator*.

For this purpose we define for any point $d \in bd(D)$ the procedure *Tangent* (d, C) as the process of identifying

- the clockwise tangent with respect to C passing through d

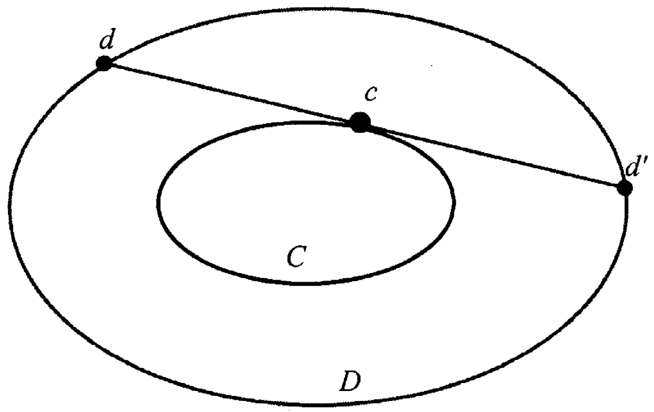

Figure 3. Example for Tangent (d, C).

- the point $c \in C$ where the tangent touches C
- the second point $d' \in bd(D)$ contained in the tangent.

Output of Tangent (d, C) is the line segment $[d, d']$ and the touching point c (see Figure 3). The following algorithm will iteratively apply the procedure Tangent (d, C) until a convex separator is found.

Greedy Algorithm for finding $cs(D \setminus C)$ **(see Figure 4)**

1. Choose $d_1 \in D$ and apply Tangent (d_1, C) to obtain $[d_1, d_2]$ and c_1, set $i = 2$.
2. Apply Tangent (d_i, C) to obtain $[d_i, d_{i+1}]$ and c_i.
3. **If** d_1 is visible from d_{i+1} choose in $[d_i, d_{i+1}]$ the point $\tilde{d}$ closest to d_i which is visible from d_1 set $d_{i+1} = \tilde{d}$ and output $bd(cs(C \setminus D)) = (d_1, c_1, d_2, \ldots, d_i, c_i, d_{i+1}, c_{i+1}, d_1)$
   Else: $i := i + 1$ and Goto 2

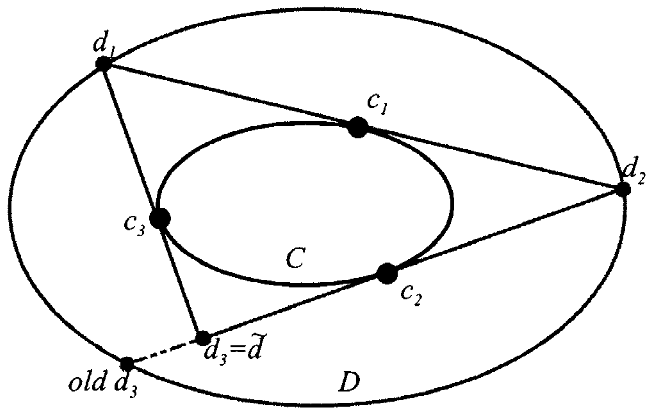

Figure 4. Example for the Greedy Algorithm.

By definition, the Greedy algorithm produces a convex separator with respect to C and D. The next theorem shows, that it is, for the purpose of applying it to the location problem 2.4, particularly well suited.

THEOREM 5. *The Greedy algorithm outputs a minimum convex separator with respect to C and D or contains one more vertex than a minimum convex separator.*

Proof. Let $cs(D \setminus C, d_1)$ be the convex separator defined by the Greedy procedure obtained from starting point d_1. Obviously the following property holds:

> If d_1 is moved clockwise along $bd(D)$ then $c_1, c_2, \ldots, c_{i+1}$ and $d_2, \ldots, d_{i+1}$ will also move clockwise along C and D, respectively.

We now show that d_1 can be chosen in such way that $cs\ (D \setminus C, d_1)$ is even a minimum convex separator.

For this purpose let the close polygon P be any minimal convex separator. Wlog we assume that every edge of P is tangent to D. (If this is not the case, move a non-tangent edge inwards along its adjacent edges until it becomes tangent (see Figure 5). This process does not increase the number of vertices of P.) Two cases may exist.

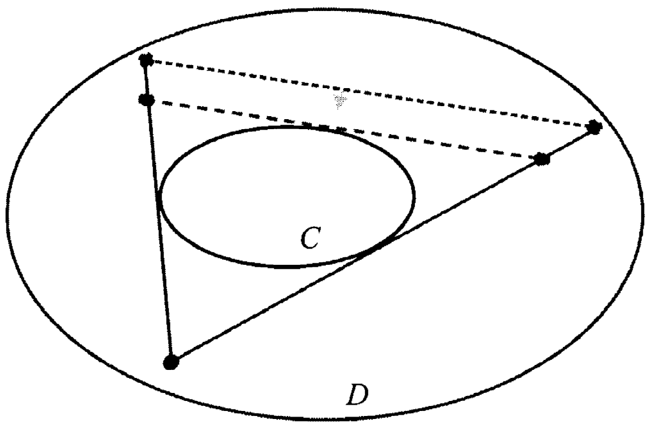

Figure 5. Moving an edge inwards.

Case 1. P contains exactly one or no vertex in int D. Then $P = cs(D \setminus C, d_1)$, where d_1 is the vertex of P clockwise next to the vertex in int C (if such a vertex exists) or any vertex of P, respectively.

Case 2. If P contains at least two vertices in int D. Let v be one of them with adjacent edges e and f in clockwise order, and let u be the other end vertex of f. If we move along the extension of e to $bd(D)$ and maintain the tangent property of f, vertex u moves clockwise along $bd(C)$. (see Figure 6)

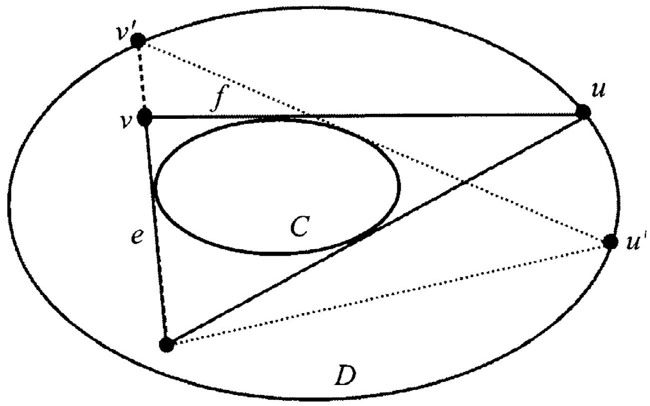

Figure 6. Moving v to the boundary.

According to the observation at the beginning of the proof all subsequent vertices of P will move clockwise along $bd(D)$ until the first node in D is reached. A new vertex in int (D) is generated resulting in a new convex separator P with the same number of vertices, but containing one more of them on $bd(D)$ than before.

By iteratively applying this procedure the assumption of Case 1 finally holds such that $cs\ (D \setminus C, d_1)$ is, indeed, a minimal convex separator.

Now let $P(D \setminus C, d_1)$ and $P(C \setminus D, b)$ be an arbitrary and minimal convex separator, respectively, both delivered by the Greedy procedure.

If the two vertices of $P(D \setminus C, b) \cap bd(D)$ next to d_1 - say $\tilde{b}$ and b' - are directly connected by an edge of $P(D \setminus C, b)$ (see Figure 7), then draw both tangents with respect to C passing through D and intersecting $P(D \setminus C, b)$ at w and v, respectively. Move v along the tangent away from d_1 until it reaches $bd(C)$ thus rotating the following edges as discussed before. This operation gives us the polygonal curve $P(D \setminus C, c_1)$ which – by construction – is a convex separator and has a number of vertices at most one larger than the number of vertices of the minimum convex separator $P(C \setminus D, b)$.

Note that $P(D \setminus C, d_1)$ is even a minimum convex separator if $d_1 = b$ or $d_1 = b'$.

If $\tilde{b}$ and b' are connected by two edges of $P(D \setminus C, b)$ the same procedure leads again to $P = (D \setminus C, d_1)$ with $|V(P(D \setminus C, d_1))| = |V(P(D \setminus C, b))| + 1$ (see Figure 8)

In the case, where the sets D and C are unit balls of the Euclidean metric, the choice of the starting point is because of the symmetry of C and D irrelevant. The proof of the preceding theorem thus implies that the following result holds.

COROLLARY 6. *If* $D = \{\underline{x} \in \mathbb{R}^2 : \|x\| \leqslant 1\}$ *is the* l_2*-unit ball and* $C = \Delta \cdot D$ *for* $\Delta > 1$, *then any* $P(C \setminus D, c_1)$ *produced by the Greedy procedure is optimal.*

Notice that in the case of Corollary 6, the location problem with polyhedral gauges may be further simplified. If the optimal numbers V of extreme points in the convex separator is known from the application of the Greedy algorithm, the

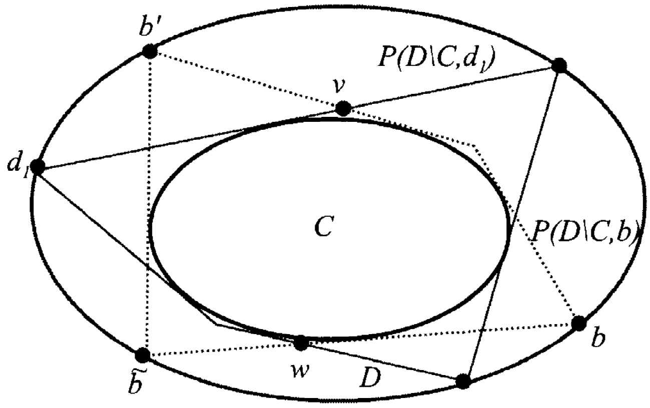

Figure 7. $\tilde{b}$ and b' are directly connected by an edge.

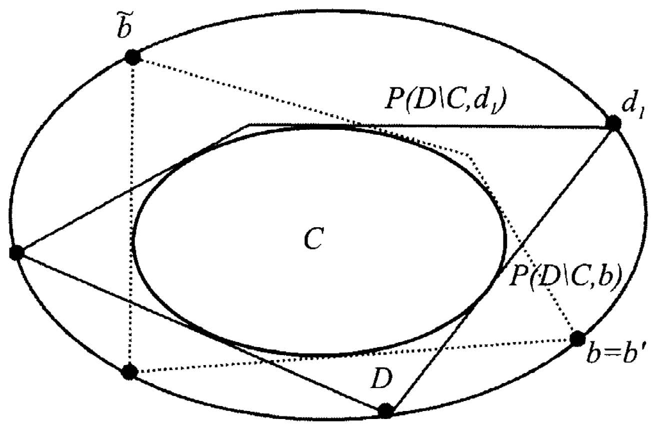

Figure 8. $\tilde{b}$ and b' are not directly connected by an edge.

Greedy convex separators may be replaced by regular V-g ones. Consequently, the usually irregular tessellation of the plane (see Figure 1) is replaced by a regular one which opens up new possibilities to improve the average running time of the algorithm.

6. Conclusion and future research topics

In this paper we have developed a polynomial approximation scheme for a very general class of location problems. The characteristic of the solution approach is the reduction of the original problem to problems in which the distance between new and existing facilities is measured by a polyhedral gauge. This modified problem can be solved by identifying a finite dominating set (FDS) of a size which is – for fixed accuracy ε – polynomial in the input of the problem.

We have presented two alternative approaches to find a suitable transformation to a polyhedral gauge problem, one based on the sandwich approach, the other on a Greedy procedure.

The algorithms presented in this paper are for some of the specific choices of feasible sets S and function Φ the only known approaches to solve these problems in a systematic way and with an *a priori* knowledge of the accuracy obtained after a given number of elementary operations. Besides the fact, that this allows the treatment of problems which so far could not be dealt with, it will also be investigated in the future, how the approach compares with alternatives in cases, where algorithms which have worked in the past satisfactorily are already available.

A first example will be problems with Euclidean distances. Here, the approximation uses polyhedral gauges with unit balls having the smallest number of extreme points. Since the unit balls can be chosen as regular V-gones the search in the resulting regular grid can be streamlined. It remains to be seen, whether the resulting algorithm will be competitive with current approaches to Euclidean location problems with non-convex objectives.

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